

MTH250 - CALCULUS

Lecture No. 26

DR. ABDUL WAHAB

Assistant Professor,
Department of Mathematics, CIIT, Pakistan.

© **VIRTUAL CAMPUS**

COMSATS Institute of Information Technology
CIIT, Virtual Campus Secretariat 402, Street 34, I-8/2 Islamabad-Pakistan.
Ph: +92 51 9101237-39 | info@vcomsats.edu.pk | vcomsats.edu.pk

These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by H. Anton, I Bivens and S. Davis, John Wiley & Sons, 10th edition, 2012.

Contents

Contents	i
26 Transformations and directional derivatives	1
26.1 Coordinate transformations	1
26.1.1 Polar coordinates	1
26.1.2 Rectangular coordinates	2
26.1.3 Cylindrical coordinates	3
26.1.4 Spherical coordinates	4
26.2 Directional derivatives	7
26.2.1 Gradients	9

LECTURE 26

Transformations and directional derivatives

26.1 Coordinate transformations

26.1.1 Polar coordinates

Definition 26.1 (Polar coordinates).

Polar Coordinates are two values that locate a point on a plane by its distance from a

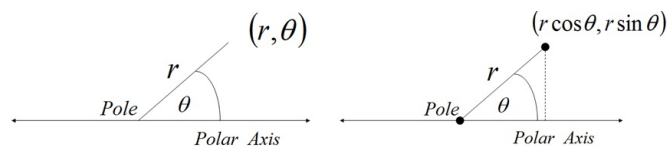


Figure 26.1: Polar coordinates.

fixed pole and its angle from a fixed line passing through the pole.

Let (x, y) be a point in xy -plane. Then using trigonometry we have

$$x = r \cos \theta, \quad y = r \sin \theta$$

where

$$r \geq 0, \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

Let (r, θ) be a point in $r\theta$ -polar coordinate plane. Again, by using trigonometry we have

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \implies \theta = \tan^{-1} \left(\frac{y}{x} \right).$$

Example 26.1. Consider the point $\left(4, \frac{\pi}{6}\right)$ in $r\theta$ -plane. In Cartesian coordinates

$$x = r \cos \theta = 4 \cos \left(\frac{\pi}{6} \right) = 4 \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

and

$$y = r \sin \theta = 4 \sin \left(\frac{\pi}{6} \right) = 4 \frac{1}{2} = 2.$$

So the point $\left(4, \frac{\pi}{6}\right)$ in Cartesian coordinates becomes $(2\sqrt{3}, 2)$.

Similarly, the point $(-5, 12)$ in xy -plane can be converted to $r\theta$ -plane as

$$\tan \theta = \frac{12}{-5} \implies \tan^{-1} \left(\frac{12}{-5} \right) \simeq -67.4^\circ + 180^\circ = 112.6^\circ.$$

and

$$r = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

Therefore, $(-5, 12)$ in polar coordinates can be written as $(13, 112.6^\circ)$.

26.1.2 Rectangular coordinates

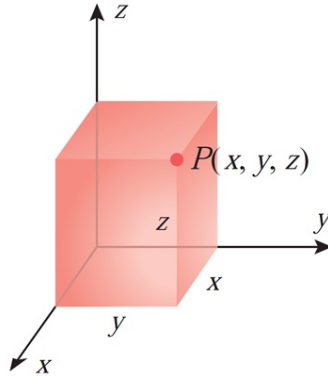


Figure 26.2: Rectangular coordinates.

Three coordinates are required to establish the location of a point in $\mathbb{R}^3$. This we can do using the rectangular coordinates (x, y, z) of a point P , where x , y and z are respectively the displacements along x -, y - and z -axis respectively; see Figure 26.2. Moreover, the coordinates can be any real numbers, without any restriction.

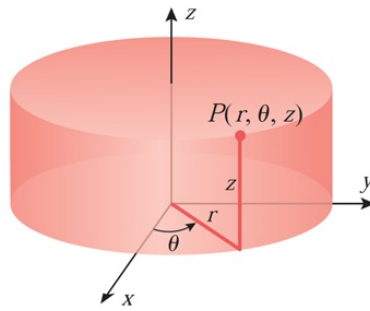


Figure 26.3: Cylindrical coordinates coordinates.

26.1.3 Cylindrical coordinates

A point in $\mathbb{R}^3$ can be represented by three quantities (r, θ, z) . These quantities are defined as follows

- r : Distance from the origin.
- θ : Angle with the polar-axis.
- z : Height above the xy - plane.

This coordinate system is called the cylindrical coordinate system. There are restrictions on the allowable values of the coordinates. We impose the following restrictions on the coordinates in cylindrical coordinate system:

$$r \geq 0, \quad 0 \leq \theta < 2\pi, \quad z \in \mathbb{R}.$$

Remark that the cylindrical coordinates just add a z -coordinate to the the polar coordinates (r, θ) , see Figure 26.4.

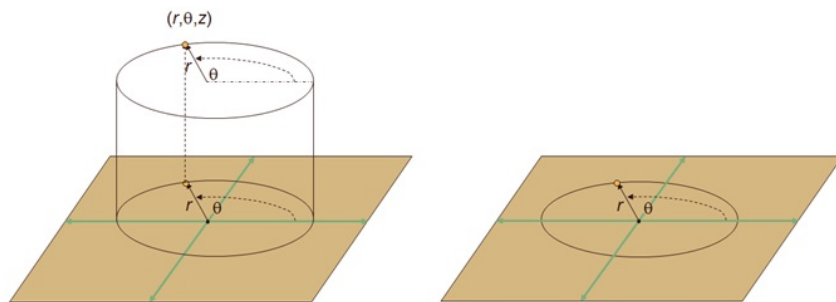


Figure 26.4: Cylindrical coordinates coordinates, analogy with polar coordinates.

Consider a point $P(r, \theta, z)$, in cylindrical coordinates. Then, in rectangular coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

Similarly, the point $P(x, y, z)$ in (r, θ, z) is given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z.$$

Example 26.2. Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = (4, \pi/3, -3).$$

Solution.

Applying the cylindrical-to-rectangular conversion formulae yields

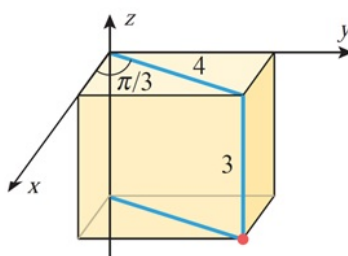


Figure 26.5: Cylindrical: $(4, \pi/3, -3)$. Rectangular: $(2, 2\sqrt{3}, -3)$.

$$x = 4 \cos\left(\frac{\pi}{3}\right) = 2, \quad y = 4 \sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}, \quad z = -3.$$

Thus, the rectangular coordinates of the point are

$$(x, y, z) = (2, 2\sqrt{3}, -3).$$

□

26.1.4 Spherical coordinates

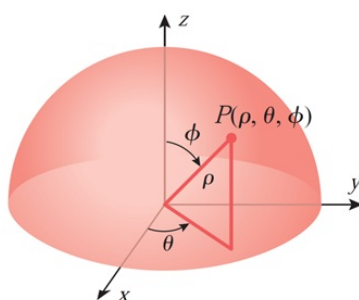


Figure 26.6: Spherical coordinates coordinates: $\rho \geq 0, 0 \leq \theta < 2\pi, 0 \leq \phi < \pi$.

A point in $\mathbb{R}^3$ can be represented by three quantities (ρ, θ, ϕ) . These quantities are defined as follows

- ρ : Distance from the origin.
- θ : Angle with the polar-axis.
- ϕ : Angle with the z -axis.

This coordinate system is called the spherical coordinate system. There are restrictions on the allowable values of the coordinates. We impose the following restrictions on the coordinates in spherical coordinate system:

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi < \pi.$$

Consider a point $P(r, \theta, \phi)$, in spherical coordinates. Then, in rectangular coordinates

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

Similarly, the point $P(x, y, z)$ in (r, θ, ϕ) is given by

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad \phi = \cos^{-1} \frac{z}{\rho}.$$

Example 26.3. Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \phi) = (4, \pi/3, \pi/4).$$

Solution.

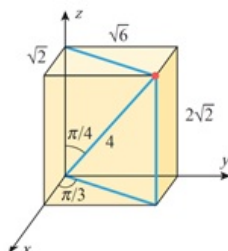


Figure 26.7: Spherical: $(4, \pi/4, \pi/3)$. Rectangular: $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

Applying the spherical-to-rectangular conversion formulae yields

$$\begin{aligned} x &= 4 \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{3}\right) = \sqrt{2}, \\ y &= 4 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{3}\right) = \sqrt{6}, \\ z &= 4 \cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}. \end{aligned}$$

The rectangular coordinates of the point are

$$(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2}).$$

□

Example 26.4. Find the spherical coordinates of the point with rectangular coordinates

$$(x, y, z) = (4, -4, 4\sqrt{6}).$$

Solution.

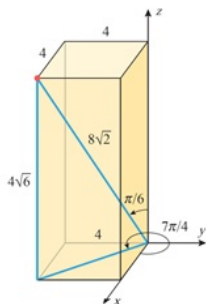


Figure 26.8: Spherical: $(8\sqrt{2}, 7\pi/4, \pi/6)$. Rectangular: $(4, -4, 4\sqrt{6})$.

From the rectangular-to-spherical-conversion formulae yields

$$\begin{aligned}\rho &= \sqrt{16 + 16 + 96} = \sqrt{128} = 8\sqrt{2}, \\ \tan \theta &= \frac{y}{x} = -1, \\ \cos \phi &= \frac{z}{\sqrt{\rho}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}.\end{aligned}$$

From the restriction $0 \leq \theta < 2\pi$ and the computed value of $\tan \theta$, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has negative y -coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \leq \phi \leq \pi$ and the computed value of $\cos \phi$, the only possibility for ϕ is $\phi = \pi/6$. Thus, the spherical coordinates of the point are

$$(\rho, \theta, \phi) = (8\sqrt{2}, 7\pi/4, \pi/6).$$

□

Example 26.5. Find equations of the paraboloid $z = x^2 + y^2$ in cylindrical and spherical coordinates.

Solution.

The rectangular-to-cylindrical conversion formulae yield $z = r^2$, which is the equation in cylindrical coordinates. Now applying the spherical-to-cylindrical conversion formulae to yield $\rho \cos \phi = \rho^2 \sin^2 \phi$ which we can rewrite as $\rho = \cos \phi \csc^2 \phi$. □

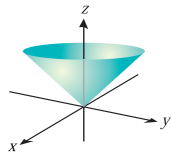
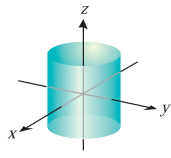
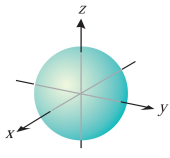
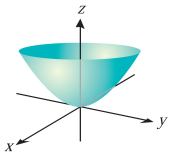
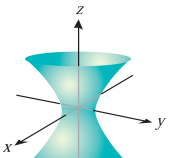
CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
				
$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
$\phi = \pi/4$	$\rho = \csc \phi$	$\rho = 1$	$\rho = \cos \phi \csc^2 \phi$	$\rho^2 = -\sec 2\phi$

Figure 26.9: Equations of surfaces in different coordinate systems: First row: Rectangular coordinates, Second row: Cylindrical coordinates, Third row: Spherical coordinates.

CONVERSION	FORMULAS	RESTRICTIONS
Cylindrical to rectangular Rectangular to cylindrical	$(r, \theta, z) \rightarrow (x, y, z)$ $(x, y, z) \rightarrow (r, \theta, z)$ $x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$ $r = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x, \quad z = z$	$r \geq 0, \rho \geq 0$ $0 \leq \theta < 2\pi$ $0 \leq \phi \leq \pi$
Spherical to cylindrical Cylindrical to spherical	$(\rho, \theta, \phi) \rightarrow (r, \theta, z)$ $(r, \theta, z) \rightarrow (\rho, \theta, \phi)$ $r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi$ $\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \tan \phi = r/z$	
Spherical to rectangular Rectangular to spherical	$(\rho, \theta, \phi) \rightarrow (x, y, z)$ $(x, y, z) \rightarrow (\rho, \theta, \phi)$ $x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$ $\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = y/x, \quad \cos \phi = z/\sqrt{x^2 + y^2 + z^2}$	

Figure 26.10: Conversion formulae for coordinate systems

26.2 Directional derivatives

Definition 26.2. If $f(x, y)$ is a function f of x and y , and if $\mathbf{u} = u_1 \hat{i} + u_2 \hat{j}$ is a unit vector, then the directional derivative of f in the direction of $\mathbf{u}$ at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{ds} [f(x_0 + su_1, y_0 + su_2)]_{s=0}$$

provided this derivative exists.

Example 26.6. Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}.$$

Solution.

It follows that

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds} \left[f \left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2} \right) \right]_{s=0}$$

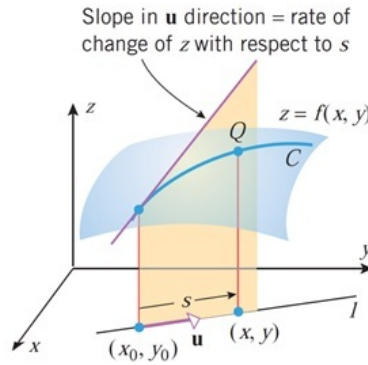


Figure 26.11: Idea of a directional derivative.

Since

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(1 + \frac{\sqrt{3}s}{2}\right) + 2,$$

we have

$$D_u f(1, 2) = \frac{d}{ds} \left[\frac{\sqrt{3}}{4}s^2 + \left(1 + \frac{\sqrt{3}s}{2}\right) + 2 \right]_{s=0} = \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3} \right]_{s=0} = \frac{1}{2} + \sqrt{3}.$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved. $\square$

Theorem 26.3. If $f(x, y)$ is differentiable at (x_0, y_0) and if $\mathbf{u} = u_1 \hat{i} + u_2 \hat{j}$ is a unit vector, then the directional derivative $D_u f(x_0, y_0)$ exists and is given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0) and if $\mathbf{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$ is a unit vector, then the directional derivative $D_u f(x_0, y_0, z_0)$ exists and is given by

$$D_u f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3.$$

Example 26.7. Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x -axis.

Solution.

The partial derivatives of f are

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}.$$

At point $(-2, 0)$ we have

$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2.$$

The unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x -axis is

$$\mathbf{u} = \cos(\pi/3)\hat{i} + \sin(\pi/3)\hat{j} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}.$$

Thus,

$$D_{\mathbf{u}}f(-2, 0) = f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) = 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3}.$$

□

26.2.1 Gradients

Definition 26.4 (Nabla operator).

The vector operator ∇ defined by

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

is called the nabla or del operator.

Definition 26.5 (Gradient).

If f is a function of x , y and z , then the gradient of f is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\hat{i} + f_y(x, y, z)\hat{j} + f_z(x, y, z)\hat{k}.$$

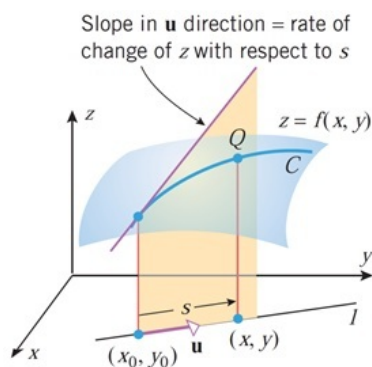


Figure 26.12: Interpretation of the gradient of a function $f(x, y)$.

Remark 26.6. *Note that by the definition of a direction derivative we have*

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u}.$$

Theorem 26.7 (Properties of gradient).

- $\nabla(u \pm v) = \nabla u \pm \nabla v$,
- $\nabla(uv) = \nabla u \nabla v + v \nabla u$,
- $\nabla v^n = n v^{n-1} \nabla v$.

Theorem 26.8. Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P .

1. If $\nabla f = \mathbf{0}$ at P , then all directional derivatives of f at P are zero.
2. If $\nabla f \neq \mathbf{0}$ at P , then among all possible directional derivatives of f at P , the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $|\nabla f|$ at P .
3. If $\nabla f \neq \mathbf{0}$ at P , then among all possible directional derivatives of f at P , the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-|\nabla f|$ at P .

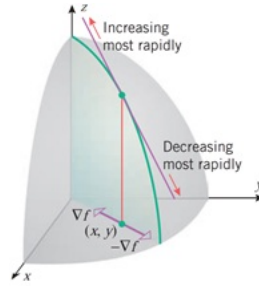


Figure 26.13: Physical interpretation of gradient.

Example 26.8. Let $f(x, y) = x^2 e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution.

Since

$$\nabla f(x, y) = f_x(x, y)\hat{i} + f_y(x, y)\hat{j} = 2xe^y\hat{i} + x^2e^y\hat{j},$$

the gradient of f at $(-2, 0)$ is $\nabla f = -4\hat{i} + 4\hat{j}$ the maximum value of the directional derivative is

$$|\nabla f(-2, 0)| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}.$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{|\nabla f(-2, 0)|} = \frac{1}{4\sqrt{2}}(-4\hat{i} + 4\hat{j}) = \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

□

Remark 26.9. Let (x_0, y_0) be a point on a level curve $f(x, y) = c$ and the curve can be smoothly parametrized as $x = x(s)$, $y = y(s)$. Then the tangent vector is

$$T(s) = x'(s)\hat{i} + y'(s)\hat{j}.$$

Now, differentiating the level curve we obtain

$$\frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} = 0 = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \right) \cdot \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} \right)$$

Therefore, T gives a direction along which f is nearly constant and so

$$D_T f(x, y) = \nabla f(x, y) \cdot T(s) = 0.$$

Theorem 26.10. Assume that $f(x, y)$ has continuous first-order partial derivatives in an open disk centered at (x_0, y_0) and that $\nabla f(x_0, y_0) \neq 0$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0) .

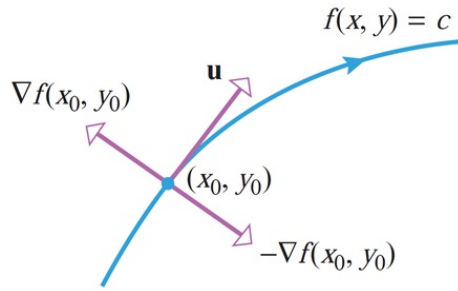


Figure 26.14: Gradient is normal to the level surface.

Remark 26.11. If u is tangent to the level curve at (x_0, y_0) , then $f(x, y)$ is neither increasing nor decreasing in that direction, so $D_u f(x_0, y_0) = 0$. Thus,

$$\nabla f(x_0, y_0), -\nabla f(x_0, y_0)$$

and the tangent vector u mark the directions of maximum slope, minimum slope, and zero slope at a point (x_0, y_0) on a level curve; see Figure 26.14.

Example 26.9. heat-seeking particle is located at the point $(2, 3)$ on a flat metal plate whose temperature at a point (x, y) is

$$T(x, y) = 10 - 8x^2 - 2y^2.$$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

Solution. Assume that the trajectory is represented parametrically by the equations

$$x = x(t), \quad y = y(t)$$

where the particle is at the point $(2, 3)$ at time $t = 0$. Because the particle moves in the direction of maximum temperature increase, its direction of motion at time t is in the direction of the gradient of $T(x, y)$, and hence its velocity vector $v(t)$ at time t points in the direction of the gradient. Thus, there is a scalar k that depends on t such that

$$v(t) = k \nabla T(x, y)$$

from which we obtain

$$\frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} = k(-16x \hat{i} - 4y \hat{j}).$$

Equating components yields

$$\frac{dx}{dt} = -16kx, \quad \frac{dy}{dt} = -4ky$$

and dividing to eliminate k yields

$$\frac{dy}{dx} = \frac{-4ky}{-16kx} = \frac{y}{4x}$$

and hence the solution

$$y = \frac{3}{\sqrt[4]{2}} \sqrt[4]{x}.$$

□